Applications of Advanced Mathematical and Computational Methods to Atmospheric and Oceanic Problems (MCAO2003)

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Preface

There is a long history of interaction of mathematics with climate dynamics going back to the pioneering work of John von Neumann and his school in the 1950s and 1960s. More recently mathematical, statistical and numerical developments are stimulating a deepening and broadening of the central concerns of climate dynamic. The purpose of this interdisciplinary summer school was to bring together graduate students and young researchers on the one hand, and, on the other hand specialists of meteorology and oceanography, and applied mathematicians interested in geophysical fluid dynamics (GFD). The ojective was firstly to initialize or further develop the communication and interactions between specialists of different fields working on diverse frontiers of GFD, and to discuss new ideas and methods that will advance the field in the next decade.
The second objective was to equip the students in these fields with the necessary tools and to bring them to the frontiers of this challenging and important area. In particular to expose advanced students and young researchers in one field to basic concepts and tools of the other field, and as well to advanced developments in their own field.
The summer school consisted of background pedagogical lectures, invited lectures and informal discussions. The complete list of courses, lectures and seminar is given below.
This volume contains the lecture notes provided by the lecturers or based on notes taken by the students. We believe that this volume will be a useful to those who attended and to those who could not attend.
The coordinators want to thank all those who contributed to the organization and to the success of this school, the speakers and the participants, and of course the staff at NCAR, Barbara Hansford, Judy Miller, Scott Briggs and Barbara Petruzzi for the proceedings. We thank also the sponsoring institutions, the Division of Mathematical Sciences and the Geoscience Division of the National Science Fundation (NSF), the Institute for Mathematics and Aplications (IMA), the National Center for Atmospheric Research (NCAR), and Indiana University.

Roger Temam, Joe Tribbia, and Shouhong Wang, Coordinators

List of Lectures

Background pedagogical lectures and Advanced Seminars

Mathematical methods:

- Instability and successive bifurcations in GFD- M. Ghil and S. Wang: Basic dynamical systems theory, stability and bifurcation analysis

- Functional analysis- R. Temam: Basic functional analysis, relations with conservation of energy and stability issues. Infinite dimensional dynamical systems theory (crucial for putting geophysical models in the perspective of infinite dimensional dynamical systems).
- Stochastic mathematics applied to the ocean and atmosphere- C. Penland and B. Ewald

Computational methods:

- Computational and numerical methods in atmoshphere and ocean – Phillip Rasch and Steve Thomas
- Numerical methods and closure Implicit methods- L. Margolin: Relations and applications in atmosphere/ocean models
- Numerical studies in the context of convection in the atmosphere - P. Smolarkiewicz

Science problems:

- Atmospheric basics and atmosphere balances- J. Tribbia
- Hamiltonian geophysical fluid dynamics - - Ted Shepherd
- Oceanography basics and turbulence- G. Vallis
- Rotating stratified turbulence in GFD- J. McWilliams
- 3D turbulence- A. Pouquet
- Lagrangian averaging and closure - D. Holm
- Large scale ocean circulation instability of time dependent flows- R. Samelson

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Differential forms and vorticity theorems

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Abstract

It is known, but not widely appreciated, that several fundamental vorticity theorems of fluid mechanics may be conveniently and naturally expressed using differential forms. The intent of this pedagogical note is to provide a concise, self-contained, accessible introduction to this appealing formulation of the classical theorems.

1 Introduction

The calculus of differential forms (Spivak, 1965) is useful for the study of geometric and topological invariants of smooth transformations. In fluid mechanics, the intrinsically kinematic or "transformation-geometric" aspects of vorticity dynamics have long been recognized (Truesdell, 1954), but the use of differential forms has been limited. The purpose of this brief pedagogical note is to illustrate how several classical vorticity theorems can be conveniently and naturally expressed in terms of differential forms. In spirit, the presentation might be seen to follow, for example, Burke's (1985) text on differential geometry for applied physicists. None of the results are new, but we hope that this introduction will help to make this elegant and appealing formulation of the classical theorems accessible to a wider community. We make use of a Hamiltonian formulation of the fluid equations that is less abstract than that of Holm *et al.* (2002), and similar, for example, to those of Salmon (1998) and Shepherd (1990). Except for the use of differential forms, the present derivation of the conservation laws from symmetries of the Lagrangian largely follows Salmon (1998). A related treatment, which includes some additional mathematical ideas and notation but omits mention of the potential vorticity (a property of particular importance in geophysical fluid dynamics), appears in Abraham *et al.* (1983).

2 Fluid motion

Let the fluid motion be defined by the map

$$x = X(a, \tau), \quad X(a, 0) = a, \tag{2.1}$$

where $X = (X_1, X_2, X_3)$ is the location at time τ of the fluid element initially at $x = a$, and $a = (a_1, a_2, a_3)$ is the Lagrangian label. The inverse map

$$a = A(x, \tau), \quad A(x, 0) = x, \tag{2.2}$$

gives the Lagrangian label of the fluid element at x at time τ. The velocity of the fluid element initially at $x = a$ is

$$u(a, \tau) = \frac{\partial}{\partial \tau} X(a, \tau), \tag{2.3}$$

and the velocity of the fluid element at x at time $\tau = t$ is

$$v(x,t) = u[A(x,t),t]. \tag{2.4}$$

Similarly, $u(a,\tau) = v[X(a,\tau),\tau]$. Let α be the Jacobian determinant of the motion,

$$\alpha = \frac{\partial(X)}{\partial(a)} = \det\left|\frac{\partial X_i}{\partial a_j}\right|. \tag{2.5}$$

It is convenient to choose the Lagrangian coordinates a so that equal volumes in a-space correspond to equal masses of fluid. Then conservation of mass for any moving volume $R(t)$ of fluid means that the integral of the density ρ over $R(t)$ is constant,

$$\int_{R(0)} d\Omega_a = \int_{R(t)} \rho(x,t)\, d\Omega_x = \int_{R(0)} \rho[X(a,\tau),\tau]\, \alpha\, d\Omega_a, \tag{2.6}$$

so the density $\rho = 1/\alpha$, and α will be the specific volume.

3 Circulation and Stokes' theorem

The circulation

$$\Gamma = \oint_{C(t)} v(x,t) \cdot dx \tag{3.1}$$

is the (clockwise-oriented) line integral at time t of the fluid velocity around the closed curve $C(t)$. With the definition of the 1-form

$$V(x,t) = v(x,t) \cdot dx = v_1(x,t)\, dx_1 + v_2(x,t)\, dx_2 + v_3(x,t)\, dx_3, \tag{3.2}$$

the circulation may be written

$$\Gamma(t) = \oint_{C(t)} V(x,t). \tag{3.3}$$

The exterior derivative dV of V is

$$dV = \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) dx_1 \wedge dx_2 + \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right) dx_2 \wedge dx_3 + \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right) dx_3 \wedge dx_1, \tag{3.4}$$

or, in terms of the components of the vorticity $\omega = \nabla_x \times v$,

$$dV = \omega_3\, dx_1 \wedge dx_2 + \omega_1\, dx_2 \wedge dx_3 + \omega_2\, dx_3 \wedge dx_1. \tag{3.5}$$

Here and below, we regard τ and t as parameters, and compute exterior derivatives only with regard to the spatial coordinates a and x; we assume also that all functions are sufficiently smooth that the derivatives make sense. The wedge products $dx_i \wedge dx_j,\ i,j = 1,2,3$, in (3.4) and (3.5) are differential 2-forms that correspond to the area elements $dx_i\, dx_j$ in the standard notation of ordinary multi-variate calculus.

Stokes' theorem (e.g, Spivak, 1965) then implies

$$\int_S dV = \oint_{C(t)} V, \tag{3.6}$$

where the surface S is bounded by the closed curve $C(t)$.

4 The pull-back

The fluid motion may be inverted to express the circulation as an integral in a-space, using the chain rule to write $dx_j = \nabla_a X_j \cdot da$, $j = 1,2,3$. This yields

$$\Gamma(\tau) = \oint_{C(0)} U(a,\tau), \tag{4.1}$$

where now the 1-form $U(a,\tau)$ is

$$\begin{aligned} U(a,\tau) &= \hat{u}\cdot da & (4.2)\\ &= u(a,\tau)\cdot\frac{\partial X(a,\tau)}{\partial a_1}\,da_1 + u(a,\tau)\cdot\frac{\partial X(a,\tau)}{\partial a_2}\,da_2 + u(a,\tau)\cdot\frac{\partial X(a,\tau)}{\partial a_3}\,da_3. & (4.3)\end{aligned}$$

In the notation and language of differential forms, $U = X^*V$, and U is the "pull-back" of V to a under the map X. If $\tau = 0$, X is the identity map, $\partial X_i/\partial a_j = \delta_{ij}$, and $U = V$. Similarly, the volume elements $d\Omega_x$ and $d\Omega_a$ in (2.6) are differential 3-forms,

$$d\Omega_x = dx_1 \wedge dx_2 \wedge dx_3, \qquad d\Omega_a = da_1 \wedge da_2 \wedge da_3, \tag{4.4}$$

and these satisfy $X^*(d\Omega_x) = \alpha d\Omega_a$. These differential 3-forms correspond to the volume elements $dx_1\,dx_2\,dx_3$ and $da_1\,da_2\,da_3$ in the standard notation of ordinary multi-variate calculus.

5 Potential vorticity: the parcel-exchange symmetry

For adiabatic ($\partial\eta/\partial\tau = 0$) motion of a thermodynamic fluid with internal energy E a function of entropy η and density $\rho = 1/\alpha$, the action integral of Hamilton's principle may be written

$$\mathcal{L} = \int_{\tau_1}^{\tau_2} L^a(u, E, \phi, \eta, X)\, d\tau, \tag{5.1}$$

where

$$L^a(u, E, \phi, \eta, X) = \int \{\frac{1}{2}u^2(a, \tau) - E[\alpha(X, a), \eta(a)] - \phi(X)\}\, d\Omega_a. \tag{5.2}$$

Here ϕ is the potential for the external force per unit mass, and the integral is over the entire volume of fluid.

Transforming L^a to x-space, we have $L^a = L^x$, where

$$L^x(v, E, \phi, \eta, A) = \int \{\frac{1}{2}v^2(x, t) - E[\alpha(x, A), \eta(A)] - \phi(x)\}\, \alpha^{-1}\, d\Omega_x. \tag{5.3}$$

For fixed v, E, ϕ, η, consider an infinitesimal re-arrangement of the initial fluid elements, $A \to A' = A + s\xi$, where s is a scalar. The first variation of L^x under this parcel-exchange operation is

$$\delta L^x \cdot \xi = \lim_{s \to 0} \frac{1}{s}[L^x(v, E, \phi, \eta, A + s\xi) - L^x(v, E, \phi, \eta, A)]. \tag{5.4}$$

If $A \to A'$ is volume-preserving, so that

$$\nabla_a \cdot \xi = 0, \tag{5.5}$$

then

$$\lim_{s \to 0} \frac{1}{s}(\alpha(x, A') - \alpha(x, A)) = \alpha(x, A) \lim_{s \to 0} \frac{1}{s}\left(\frac{\partial(A')}{\partial(A)} - 1\right) = \alpha(x, A)\, \nabla_a \cdot \xi = 0, \tag{5.6}$$

and if $A \to A'$ is also adiabatic, so that

$$\nabla_a \eta \cdot \xi = 0, \tag{5.7}$$

then

$$\delta L^x \cdot \xi = 0. \tag{5.8}$$

That is, the Lagrangian L^x is invariant under an arbitrary, infinitesimal, volume-preserving, adiabatic, parcel-exchange transformation $A \to A'$.

Now, for ξ, compute the terms in the variation of the action integral (5.1). After an integration by parts over τ on the first term in the integrand, the result may be written

$$\delta \mathcal{L} \cdot \xi = \left[\int u \cdot \xi' d\Omega_a\right]_{\tau_1}^{\tau_2} - \int_{\tau_1}^{\tau_2} \int \mathcal{E} \cdot \xi'\, d\Omega_a d\tau, \tag{5.9}$$

where

$$\mathcal{E} = \frac{\partial u}{\partial \tau} + \alpha \nabla_x p + \nabla_x \phi, \qquad p = -\frac{\partial E}{\partial \alpha}, \tag{5.10}$$

and ξ' is the variation in $X(A,\tau)$ induced by the transformation $A \to A'$, with components $\xi'_j = \nabla_a X_j \cdot \xi$, $j = 1,2,3$. Then (Noether's theorem), since $\delta\mathcal{L}\cdot\xi = \int \delta L^x \cdot \xi \, dt = 0$ (the parcel-exchange symmetry), and since $\mathcal{E} = 0$ for a fluid motion (that is, for a solution of the Euler equations),

$$\left(\int u\cdot\xi' d\Omega_a\right)(\tau=\tau_2) = \left(\int u\cdot\xi' d\Omega_a\right)(\tau=\tau_1). \tag{5.11}$$

Since τ_1 and τ_2 are arbitrary, and since $u\cdot\xi' = \hat{u}\cdot\xi$, it follows that

$$\frac{\partial}{\partial\tau}\left(\int \hat{u}\cdot\xi d\Omega_a\right) = 0 \tag{5.12}$$

The conditions (5.5) and (5.7) are satisfied if

$$\xi(a) = \nabla_a \times (h\nabla_a\eta) = \nabla_a h \times \nabla_a \eta, \tag{5.13}$$

where h is an arbitrary scalar function of a. Substitution of (5.13) in (5.12), integration by parts over a-space, and the observation that h is arbitrary yields

$$\frac{\partial}{\partial\tau}\Pi = 0, \tag{5.14}$$

where the differential 3-form Π is given by

$$\Pi = P\, da_1 \wedge da_2 \wedge da_3 = dU \wedge d\eta = d(U\wedge d\eta) = -d(\eta dU), \tag{5.15}$$

with $P(a) = \nabla_a \times \hat{u}\cdot\nabla_a\eta = \nabla_a\cdot(\hat{u}\times\nabla_a\eta) = \nabla_a\cdot(\eta\nabla_a\times\hat{u})$. In (5.15), dU is the exterior derivative of U,

$$dU = \left(\frac{\partial\hat{u}_2}{\partial a_1} - \frac{\partial\hat{u}_1}{\partial a_2}\right) da_1\wedge da_2 + \left(\frac{\partial\hat{u}_3}{\partial a_2} - \frac{\partial\hat{u}_2}{\partial a_3}\right) da_2\wedge da_3 + \left(\frac{\partial\hat{u}_1}{\partial a_3} - \frac{\partial\hat{u}_3}{\partial a_1}\right) da_3\wedge da_1, \tag{5.16}$$

and $d\eta = \nabla_a\eta\cdot da$. The equation (5.14) expresses the material conservation of isentropic potential vorticity $P(a)$. Since $d(X^*V) = X^*dV$, it follows that

$$\begin{aligned}
\Pi(0) = \Pi(\tau) &= X^*dV \wedge d\eta \\
&= X^*dV \wedge X^*\left(\sum_{k=1}^{3}\frac{\partial\eta}{\partial x_k}dx_k\right) \\
&= X^*\left(\sum_{k=1}^{3} dV \wedge \frac{\partial\eta}{\partial x_k}dx_k\right) \\
&= X^*\left(\sum_{k=1}^{3}\omega_k\frac{\partial\eta}{\partial x_k}dx_1\wedge dx_2\wedge dx_3\right) \\
&= \sum_{k=1}^{3}\omega_k[X(a,\tau),\tau]\frac{\partial\eta}{\partial x_k}\alpha da_1\wedge da_2\wedge da_3.
\end{aligned} \tag{5.17}$$

Thus, the materially conserved potential vorticity in x-space is

$$Q(x,t) = \frac{\omega}{\rho} \cdot \nabla_x \eta. \tag{5.18}$$

The last expression in (5.15) immediately gives the following surprising result (Haynes and McIntyre, 1990): the integral of Π over an a-volume (or the integral of ρQ over an x-volume) bounded by two isentropic surfaces S_1 and S_2 that each surround the entire globe is exactly zero.

6 Vorticity theorems for a homentropic fluid

Suppose the fluid is homentropic, that is, has uniform entropy. Then the parcel-exchange symmetry yields a potential vorticity conservation law for each independent Lagrangian label a_j, $j = 1, 2, 3$, and, by (5.14), each component of $\nabla_a \times \hat{u}$ is materially conserved. Using (5.16), this general vorticity conservation law (Lamb, 1932, p. 204) may be written in our notation

$$\frac{\partial}{\partial \tau}(dU) = 0. \tag{6.1}$$

Two fundamental results that follow in this case are Kelvin's circulation theorem,

$$\frac{d}{d\tau}\Gamma(\tau) = 0, \tag{6.2}$$

and Cauchy's integral of the vorticity equation,

$$\frac{\omega(X(a,\tau),\tau)}{\rho} = (\omega(a,0) \cdot \nabla_a) X(a,\tau). \tag{6.3}$$

For Kelvin's circulation theorem, we have

$$\begin{aligned} \frac{d}{d\tau}\Gamma &= \frac{d}{d\tau} \oint_{C(t=\tau)} V(x, t=\tau) \\ &= \frac{d}{d\tau} \oint_{C(0)} U(a,\tau) \\ &= \frac{d}{d\tau} \int_{R_0} dU(a,\tau) \\ &= \int_{R_0} \frac{\partial}{\partial \tau} dU(a,\tau) = 0, \end{aligned} \tag{6.4}$$

where R_0 is any surface in a-space bounded by $C(0)$.

For Cauchy's vorticity theorem, we have (since also $\partial a_j/\partial\tau = 0$, $j = 1,2,3$)

$$\begin{aligned}
\omega_j(a,0)da_1\wedge da_2\wedge da_3 &= dU\wedge da_j \\
&= X^*dV(x,t=\tau)\wedge da_j \\
&= X^*dV\wedge X^*\left(\sum_{k=1}^{3}\frac{\partial A_j}{\partial x_k}dx_k\right) \\
&= X^*\left(\sum_{k=1}^{3}\omega_k\frac{\partial A_j}{\partial x_k}dx_1\wedge dx_2\wedge dx_3\right) \\
&= \sum_{k=1}^{3}\omega_k[X(a,\tau),\tau]\frac{\partial A_j}{\partial x_k}\alpha da_1\wedge da_2\wedge da_3. \qquad (6.5)
\end{aligned}$$

Thus,

$$\frac{\omega(X(a,\tau),\tau)}{\rho}\cdot\nabla_x A_j = \omega_j(a,0). \qquad (6.6)$$

Since

$$\sum_{k=1}^{3}\frac{\partial A_i}{\partial x_k}\frac{\partial X_k}{\partial a_j} = \delta_{ij}, \qquad (6.7)$$

(6.3) follows.

From (6.1) and the Poincaré lemma, it follows upon exchanging the order of differentiation in (6.1) that in a simply-connected region $R(0)$,

$$\frac{\partial}{\partial\tau}U = dF, \qquad (6.8)$$

for some function $F(a,\tau)$. Then

$$\begin{aligned}
\frac{d}{dt}\int_{R(t)} v\cdot\omega\, d\Omega_x &= \frac{d}{dt}\int_{R(t)} V\wedge dV \\
&= \frac{d}{d\tau}\int_{R(0)} U\wedge dU \\
&= \int_{R(0)}\left(\frac{\partial}{\partial\tau}U\right)\wedge dU + \int_{R(0)} U\wedge\left(\frac{\partial}{\partial\tau}dU\right) \\
&= \int_{R(0)} dF\wedge dU \\
&= \int_{R(0)} d(F\wedge dU) \\
&= \int_{S(0)} F\, dU \\
&= \int_{S(t)} F\, dV, \qquad (6.9)
\end{aligned}$$

where the surface $S(t)$ is the boundary of the volume $R(t)$. If $S(t)$ is everywhere tangent to the vorticity ω, then $dV = 0$ on $S(t)$, and the integral of the helicity $v \cdot \omega$ over $R(t)$ is conserved following the motion. For a fluid motion, $\mathcal{E} = 0$ in (5.10), and from (6.8) and (4.2) it follows that (up to an arbitrary constant) F is the Bernoulli function,

$$F(a, \tau) = -\int^{\alpha} \tilde{\alpha} \frac{d^2 E(\tilde{\alpha})}{d\tilde{\alpha}^2} d\tilde{\alpha} - \phi + \frac{1}{2}(u \cdot u) + const. \tag{6.10}$$

The standard derivation of the vorticity theorems amounts essentially to rewriting the equations of motion $\mathcal{E} = 0$ in terms of F first, and then obtaining (6.1) from the exterior derivative of (6.8).

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