A Potential Vorticity Etymology

R. M. Samelson

Woods Hole Oceanographic Institution

Woods Hole, MA 02543

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Abstract

The hydrodynamical quantity known as the "potential vorticity" has come to play a central role in meteorology and oceanography. However, the term itself is mysterious: the quantity does not have the dimensions of vorticity, and it is not clear what is meant by "potential." A brief summary of the origin of the concept and the term is given.

1. Introduction

In the half century since its derivation by Rossby (1936) and Ertel (1942a), the hydrodynamical quantity known as the "potential vorticity" has come to play a central role in meteorology and oceanography. Varieties of this quantity are conserved along particle paths for certain general classes of fluid flow; moreover, its distribution—along with appropriate boundary conditions and the near-geostrophic balance of large-scale motion in the atmosphere and ocean—itself essentially determines the large-scale velocity field. Under the well-known quasi-geostrophic approximation, for example, the flow is governed by an equation for the time evolution of a potential vorticity, plus a diagnostic elliptic equation for the instantaneous motion field that is forced by the potential vorticity distribution.

While the concept has provided a tremendous amount of insight, the term "potential vorticity" seems slightly mysterious, or at least not entirely apt, and remains to many (students and researchers alike) the source of a certain amount of confusion. First, the quantity does not even have the dimensions of vorticity. Second, it is not clear what is meant by "potential." The standard texts (e.g., Holton, 1979; Pedlosky, 1987; Gill, 1981) are not much help, as the explanations at best suggest that the word "potential" is meant to imply a kind of reservoir (of vorticity). The review article by Hoskins *et al.* (1985) contains some of the relevant history, but sidesteps a central question: how did the term "potential vorticity," which (it turns out) originally stood for something else, come to stand for what we now call the potential vorticity? Here, a brief summary of the origin of the concept and the term is given, and an attempt is made to answer this question.

2. Rossby and Ertel

In 1936, Carl Gustav Rossby, a Swedish meteorologist who made many fundamental contributions, published the Lagrangian conservation law (Rossby, 1936)

$$f + \zeta = cD \tag{1}$$

which holds exactly along fluid parcel trajectories in flow governed by the inviscid shallow water equations in a reference frame rotating at local angular velocity $f/2$ about the local vertical (note that f may vary with position, as in the β-plane approximation; see, e.g., Pedlosky, 1987). Here ζ is the relative vorticity $v_x - u_y$ (with u and v the x- and y-components of the horizontal velocity), D is the thickness of the (homogeneous) layer of fluid, and c is a constant that may depend upon the initial position of the fluid parcel. In the course of analyzing a reduced-gravity shallow water model of the Gulf Stream, Rossby (1936) derived (1), compared the conserved quantity represented by c to the Bernoulli function in steady flow, and showed how the local relative vorticity was related to the local thickness and the initial (resting) thickness by the conservation law.

Rossby (1940) gave a more extensive discussion of the result for an atmosphere consisting of a stably-arranged set of layers of constant density (or potential temperature), and recorded the additional forms

$$\frac{d}{dt}\left(\frac{f+\zeta}{D}\right) = 0, \tag{2}$$

$$\frac{d}{dt}\left(\frac{f+\zeta}{\Delta}\right) = 0, \tag{3}$$

$$f + \zeta = c\Delta \tag{4}$$

$$\zeta = \zeta_0 + (f_0 - f) + (f_0 + \zeta_0)\frac{(D - D_0)}{D_0} \tag{5}$$

$$\zeta = \zeta_0 + (f_0 - f) + (f_0 + \zeta_0)\frac{(\Delta - \Delta_0)}{\Delta_0} \tag{6}$$

where Δ is the fluid weight between two isentropic layer interfaces, and the subscript zero indicates the value at t = 0. Rossby (1940) also argued that the results (5) and

(6) apply to a continuously-stratified fluid, if the layers are taken to be infinitesimally thin, and the derivatives with respect to x and y in the expression for ζ are evaluated holding density (or potential temperature) fixed, that is, along the isopycnal (or isentropic) surfaces.

The term "potential vorticity" was invented by Rossby (1940), but what he meant by it was not what we mean by it today:

> "...the constant c [in (1) or ((4)], the physical meaning of which is not very clear, may be replaced by the ζ_0 [in (5) or (6)]. *This quantity, which may be called the potential vorticity, represents the vorticity the air column would have if it were brought, isopycnically or isentropically, to a standard latitude (f_0) and stretched or shrunk vertically to a standard depth D_0 or weight Δ_0.*"

Rossby's choice of the term "potential vorticity" was motivated by the analogy with well-known (then and now) quantities such as potential temperature, which are similarly defined in terms of the value a parcel would have if brought adiabatically to a reference state. The early descriptive use of the concept (Starr and Neiburger, 1940) followed Rossby's nomenclature carefully, and focussed on the use of the conserved potential vorticity as a tracer.

In 1942, Hans Ertel, an Austrian meteorologist known primarily for this achievement, published a general form of the potential vorticity conservation equation for a continuously-stratified fluid, using vector notation (Ertel, 1942a, 1942b, 1942c). His equation (in slightly different notation) took the form

$$\frac{d}{dt}[\rho^{-1}(\mathbf{curl}\ \mathbf{u} + 2\Omega\mathbf{k}).\mathbf{grad}\ \psi] = 0 \tag{7}$$

for materially-conserved ψ,

$$\frac{d\psi}{dt} = 0, \quad \psi = \Psi(p, \rho). \tag{8}$$

Ertel noted the special case $\psi = \theta$, where θ is potential temperature, and even suggested (Ertel, 1942c) that it might be useful in the context of Rossby's "isentropic analysis" of atmospheric conditions, but did not cite Rossby's results (1)-(5). It is generally believed that Ertel's result was obtained independently of Rossby, and the name Ertel is often associated with the first derivation of a continuously-stratified form of the potential vorticity, which is sometimes referred to as the "Ertel potential vorticity." Both of these beliefs are This belief is evidently unfounded: not only did Rossby publish a continuously-stratified form several years earlier (Rossby, 1938), but Ertel visited the Meteorology Department at MIT for several months in 1937, while Rossby was serving as department chairman and just a year after Rossby's first published derivation of a potential vorticity conservation principle (Rossby, 1936). Although the evidence is circumstantial, it is difficult to believe that Ertel's results were not influenced in some manner by interaction with Rossby during this visit. It seems perhaps most likely that Ertel's contribution was to generalize Rossby's result. It is noteworthy that his publications appeared in German or Austrian journals during the Second World War, and that all references in those publications are to German-language journals. Political considerations may have made him reluctant to refer to Rossby's work explicitly, and the odd parenthetical reference to Rossby's "isentropic analysis" may have been a purposeful indirect acknowledgement (G. Platzman, personal communication).

Rossby's expression for the potential vorticity in the shallow water equations may be recovered (asymptotically, for small aspect ratio) from Ertel's result by taking $\psi = z/h$, where h is the fluid thickness (Pedlosky, 1987). Rossby's argument for the continuously-stratified case leads to a scalar expression that may be derived from Ertel's result by taking $\psi = \theta$ and making traditional meteorological approximations, such as the hydrostatic approximation (Hoskins *et al.*, 1985).

The first descriptive use of Ertel's result was evidently that of Kleinschmidt

(1950-1951), who simply called the conserved quantity the "Ertel quantity." He noted the connection with Rossby's result, but preferred to work from the more general Ertel expression. Hoskins *et al.* (1985) point out that Kleinschmidt's work was ahead of its time in suggesting that the instantaneous motion field depended directly on the potential vorticity distribution, rather than using the conserved quantity simply as a tracer. Note, however, that the context of Rossby's (1936) original derivation was dynamical (he suggested that as a consequence of the conservation law, countercurrents should occur along the edges of certain geostrophic jets); only later did Rossby and co-workers focus on the tracer properties of the potential vorticity (Rossby, 1940; Starr and Neiburger, 1940).

3. Charney and von Neumann

How did the quantity that we know today as the "potential vorticity" come to assume the name that had originally, and with rational motivation, been given to a different, though closely related, quantity by Rossby (1940)? It seems that the theoreticians are at fault.

In the late 1940's, Jule Charney, a young American meteorologist who was strongly influenced by Rossby's ideas, derived a reduced set of equations appropriate for large-scale atmospheric motions in the context of his work on mid-latitude cyclogenesis. Charney (1948) found that these "quasi-geostrophic" equations could be elegantly stated in terms of Rossby's principle of conservation of potential vorticity, concluding that

> "...the motion of large-scale atmospheric disturbances is governed by laws of conservation of potential temperature and absolute potential vorticity, and by the conditions that the horizontal velocity be quasi-geostrophic, and the pressure quasi-hydrostatic."

In this landmark derivation, Charney explicitly retained Rossby's nomenclature for the potential vorticity: he wrote the conservation principle as

$$\frac{(q_\theta)_1}{(\delta p)_1} = \frac{(q_\theta)_0}{(\delta p)_0} \tag{9}$$

where δp was the pressure difference between isentropic surfaces with potential temperature θ and $\theta + \delta\theta$, and q_θ was the absolute vorticity component perpendicular to the isentropic surface, observing that

> "...If we choose a standard value for $(\delta p)_0$, then $(q_\theta)_0$ is a constant of the motion, and we shall call it the absolute potential vorticity to conform to the terminology introduced by Rossby (1940)."

Charney continued to use this terminology as late as 1949, when he derived the conservation law for a continuously stratified fluid in the form

$$\frac{d}{dt}[\rho^{-1}(\mathbf{curl}\ \mathbf{u}\ + 2\Omega\mathbf{k}).\mathbf{grad}\ \sigma] = 0 \tag{10}$$

for materially-conserved σ,

$$\frac{d\sigma}{dt} = 0, \quad \sigma = S(p, \rho) \tag{11}$$

noting that for $\sigma = \theta$, the result is "...essentially Rossby's equation of conservation of 'potential vorticity' (Rossby, 1940)..." (Charney, 1949). This, of course, is exactly the form (7) that had been derived by Ertel (1942a) some seven years earlier. It is remarkable that (in a footnote) Charney (1949) refers to two other papers by Ertel, on the impossibility of limited-area dynamical weather predictions, but not to Ertel's previous derivation of the conservation law! (Rossby and Ertel must have been aware of each other's results by 1949, as they co-authored a pair of articles (Ertel and Rossby, 1949a, 1949b), on a related vorticity invariant, which make use of and refer to Ertel (1942b, 1942c).)

Before leaving the U.S. in 1947 for his postdoctoral tenure in Norway, Charney had become aware that John von Neumann, a well-known Hungarian mathematician and physicist who spent much of his life in the U.S., was interested in the possibility of "weather forecasting by computing" using the ENIAC, the world's first digital computer, which had been developed by J. Presper Eckert and John W. Mauchly at Pennsylvania State University. Von Neumann was interested in the "adequate filtering method for treating the vertical velocities in the equations of atmospheric dynamics" that Charney had evidently found, and invited Charney to join the forecasting project at Princeton following his return from Norway. In August, 1948, Charney sent von Neumann (then at Los Alamos, New Mexico) a 23-page typewritten letter that detailed several difficulties with using the hydrodynamic equations directly for computational forecasts, outlined the derivation of an "improved system of equations" (the quasi-geostrophic equations) and a method for their numerical solution, and proposed a multi-stage "immediate attack on the numerical forecast problem" that would begin with the numerical solution of a barotropic model. Von Neumann was not immediately convinced by Charney's reasoning, and an exchange of letters ensued, in which Charney repeated his argument that the inability to determine horizontal divergences (of the velocity or density flux) observationally made it necessary to introduce the potential vorticity equation in place of the continuity equation. The project proceeded following Charney's proposal. (The correspondence between Charney and von Neumann is preserved in the MIT archives (Jule Charney Papers, MC 184).)

For a barotropic fluid with a rigid lid, there are no variations in thickness, so by (1) the potential vorticity and the vorticity differ only by a constant factor, and the quasi-geostrophic potential vorticity equation reduces to the barotropic vorticity equation. Charney *et al.* (1950) solved the barotropic model by time-stepping evolution equations for the vorticity and the streamfunction, where the streamfunction tendency was obtained from the vorticity tendency at each step by solving a diagnostic elliptic equation. The quasi-geostrophic equations may be solved in a similar way, except that the time-stepped vorticity quantity is (what we now call) the quasi-geostrophic potential vorticity. [Note: Since the quasi-geostrophic potential vorticity is not the geostrophic approximation to the true potential vorticity, and is conserved along the projections of particle paths on horizontal planes, rather than along the particle paths themselves, an attempt was made to preserve the distinction by introducing the term "pseudo-potential vorticity" (Charney and Flierl, 1981), but this usage did not catch on.] The convenience of having the same name (vorticity) for the conserved quantity and for the evolution equation in the barotropic model must have overwhelmed the instinct to preserve the original nomenclature. Rossby himself had begun to refer to (2) as a "vorticity equation" (Rossby, 1949), although its derivation depends also on the continuity equation. By the time Charney and Norman Phillips began numerical integrations of the quasi-geostrophic equations (Charney and Phillips, 1953), the change of name had been made, without explanation or comment:

"The potential vorticity may be defined by $q = -(f + \zeta)(\partial p/\partial\theta)^{-1}$."

This evolution of the terminology may have been inevitable from the moment that Charney (1948) demonstrated the fundamental dynamical importance of the potential vorticity conservation equation. Nonetheless, one might say that, with this sentence, the modern era in geophysical fluid dynamics had begun.

4. What should it have been called?

Reviewing these events, one can't easily escape the sense that a historical chance has been missed: the generalized quantity for which Rossby and Ertel derived conservation laws has proven to deserve a name of its own, but failed to get one. Instead, it inherited a name that was intended for something else and doesn't really fit.

With this in mind, it is difficult to resist the opportunity to speculate on what the quantity might have been called, had its future importance and unfortunate lexical destiny been known to those who found it. In Rossby's original terms, the conservation law actually provides a quantity from which one can form a "potential vorticity," which leads directly to the frivolous, but nonetheless more exact, "potential potential vorticity" (M. McCartney, personal communication). One could return to Latin roots for a variant on vorticity, such as (from *virt-*, strength, and *tors-*, twist) "virtorsity" (perhaps a perilous choice for verbal presentations). Both of these, however, neglect an essential component (as does the existing term, notwithstanding the present-day reinterpretation of the word "potential"): the stratification of the fluid by the appropriate conserved scalar. Combining "stratification" and "vorticity," one naturally obtains "strativorticity," which would seem to have fit quite well.

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