Eulerian and Lagrangian means and Stokes drift – RMS 15 Nov 2023

For surface waves, Phillips (1977) shows that:

1. The depth-integrated Eulerian and Lagrangian mean momenta are equal (Phillips, 1977, p. 44);

2. The only contribution to the Eulerian-mean wave momentum arises from regions of space above the wave troughs (Phillips, 1977, p. 40).

It is straightforward to verify and illustrate these general results for the case of a linear sinusoidal deep-water wave.

Following Section 3.2 of Phillips (1977), let the free-surface displacement $z = \zeta(x,t)$ of an infinitesimal (linear) wave be:

$$\zeta(x,t) = a\cos(kx - \sigma t), \tag{0.1}$$

with the x-axis oriented in the direction of propagation of the wave and the mean free surface at $z = 0$. The horizontal velocity in the x-direction is:

$$u(x,z,t) = \sigma a e^{kz}\cos(kx - \sigma t). \tag{0.2}$$

The classical Lagrangian-mean motion (Stokes drift) for this wave can be computed by integrating the equations for parcel motion in the two-dimensional velocity field (e.g., Phillips, 1977, Section 3.3). The result, to leading order in ka, is:

$$u_S(z) = \sigma k a^2 e^{2kz}, \quad z \leq 0, \tag{0.3}$$

where z can be regarded either as the mean depth of the parcel or as the initial position of a parcel at the nodal points $x_n = \pi/2 + n\pi$, where $\zeta(x_n, t = 0) = 0$ (Fig. 1; blue line). The vertical integral of this mean parcel motion gives the depth-integrated mean Lagrangian momentum:

$$U_S = \int_{-\infty}^{0} u_S(z)\, dz = \sigma k a^2 \int_{-\infty}^{0} e^{2kz}\, dz = \frac{1}{2}\sigma a^2. \tag{0.4}$$

That the parcel-following Lagrangian-mean calculation gives the correct depth-integrated wave momentum can be checked by computing the Eulerian mean, which requires only averaging the horizontal velocity field; it is easily seen that the mean vertical velocity vanishes. This Eulerian mean (over x at $t = 0$, for simplicity) is:

$$u_E(z) = \frac{k}{2\pi} \int_{-\pi/k}^{\pi/k} u(x, z, t = 0)\, dx. \tag{0.5}$$

For $z < -a$ (below the wave troughs):

$$u_E(z < -a) = \frac{k\sigma a}{2\pi} e^{kz} \int_{-\pi/k}^{\pi/k} \cos kx\, dx = \frac{\sigma a}{2\pi} e^{kz} \sin kx|_{-\pi/k}^{\pi/k} = 0. \tag{0.6}$$

For $|z| < a$ (above the wave troughs):

$$\begin{aligned} u_E(|z| < a) &= \frac{k\sigma a}{2\pi} e^{kz} \int_{-\arccos(z/a)/k}^{\arccos(z/a)/k} \cos kx\, dx \\ &= \frac{\sigma a}{2\pi} e^{kz} \sin kx|_{-\arccos(z/a)/k}^{\arccos(z/a)/k} \\ &= \frac{\sigma a}{\pi} [1 - (z/a)^2]^{1/2}, \end{aligned} \tag{0.7}$$

where $e^k z = 1 + \mathscr{O}(ka) \approx 1$ has been used (Fig. 1; green line). This illustrates point #2 above.

The vertical integral of this Eulerian mean is:

$$
\begin{aligned}
U_E &= \int_{-\infty}^{a} u_E(z)\,dz \\
&= \frac{\sigma a^2}{\pi}\int_{-1}^{1}[1-(z/a)^2]^{1/2}\,d(z/a) \\
&= \frac{\sigma a^2}{\pi}\int_{-1}^{1}\cos\theta\,d(\sin\theta) \\
&= \frac{\sigma a^2}{\pi}\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta \\
&= \frac{1}{2}\sigma a^2. \qquad (0.8)
\end{aligned}
$$

This shows that $U_S = U_E$, confirming the accuracy of the Lagrangian-mean calculation of the depth-integrated mean wave momentum, and illustrates point #1 above.

Phillips, O. M., 1977. *The dynamics of the upper ocean.* 2nd ed., Cambridge Press, 270 pp.

Samelson, R. M., 2022. *J. Phys. Oceanogr.*, 52, 1945-1967. doi: 10.1175/JPO-D-22-0017.1.

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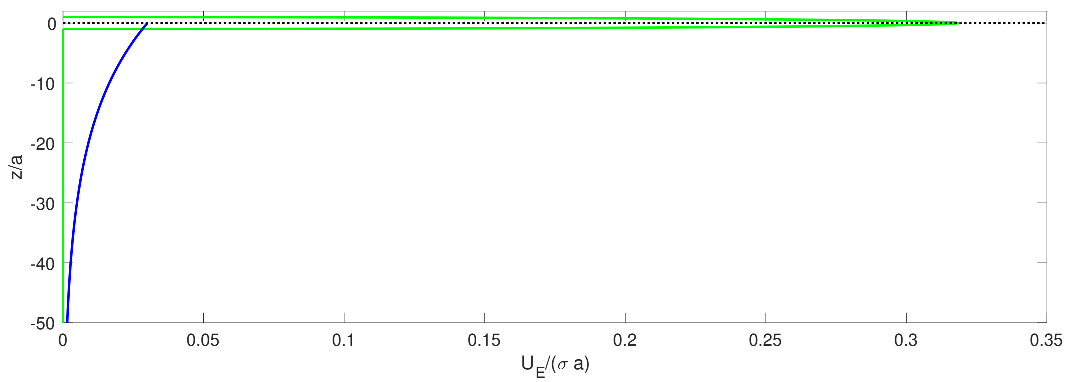

Figure 1: Dimensionless wavelength-averaged Eulerian horizontal velocity (green) and Lagrangian Stokes drift (blue) vs. dimensionless depth for a linear sinusoidal surface-gravity wave with free-surface displacement $\zeta = a\cos(kx - \sigma t)$ for $ka = 0.03$. The vertical integrals of the two profiles are equal. The Stokes-drift profile can also be obtained as a spatial mean of the Eulerian velocity by computing a mass-weighted average on surfaces of constant pseudo-vertical, surface-conforming, orthogonal curvilinear coordinate, as shown in Samelson (2022).