Periodic and Primitive Solutions to the KdV Equation and KB System

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Solution to the KdV equation $u_t - uu_x + u_{xxx} = 0$ with initial condition $u(x, 0) = \sum_n 1_{[\pi, 2\pi]}(x - 2\pi n)$ using operator splitting method:
The Lax Pair for the KdV Equation

1. Consider the system

\[ L \psi = E \psi, \quad \psi_t = A \psi \]  \hspace{1cm} (1)

where

\[ L = -\frac{\partial^2}{\partial x^2} + u(x, t), \]  \hspace{1cm} (2)

\[ A = -4 \frac{\partial^3}{\partial x^3} + 6u(x, t) \frac{\partial}{\partial x} + 3u_x(x, t). \]  \hspace{1cm} (3)

2. The compatibility condition is the Lax equation

\[ L_t = [A, L]. \]  \hspace{1cm} (4)

3. This is isospectral and equivalent to

\[ u_t - 6uu_x + u_{xxx} = 0. \]  \hspace{1cm} (5)

Lax '68
Inverse Scattering Transform for Generic Periodic Solutions to the KdV Equation

A generic real potential $u \in L^\infty(\mathbb{R})$ with $u(x + T) = u(x)$ is determined by:

- The $L^2(\mathbb{R})$ spectrum
  \[
  \sigma(L) = \bigcup_{n=0}^{\infty} [\lambda_{2n}, \lambda_{2n+1}] \tag{6}
  \]
  where $\{\lambda_j\}$ is a sequence of increasing interlaced periodic and antiperiodic eigenvalues of $L$. (WLOG $\lambda_0 = 0$)

- The Dirichlet eigenvalues $\mu_n \in (\lambda_{2n-1}, \lambda_{2n})$ of $L$ on $[0, T]$.

- The signature $\sigma_n = \text{sgn}(\log(|y_2'(T, \mu_n)|))$. 

McKean, Trubowitz '76,'78
Some Notation

Definition

\[ B(\lambda) := \sqrt{\frac{4(\lambda_0 - \lambda)}{\prod_{n=1}^{\infty} \frac{T^4}{n^4 \pi^4} (\lambda_{2n} - \lambda)(\lambda_{2n-1} - \lambda)}} \]

\[ f^\pm(\lambda) := \prod_{\sigma_n = \pm 1}^{\infty} \frac{T^2}{n^2 \pi^2} (\mu - \lambda), \quad f^0(\lambda) := \prod_{\sigma_n = 0}^{\infty} \frac{T^2}{n^2 \pi^2} (\mu - \lambda). \]

\[ V(x, t, \lambda) := \begin{cases} (-1)^{n-1} \begin{pmatrix} 0 & i \frac{f^+(\lambda)}{f^- (\lambda)} \\ i \frac{f^- (\lambda)}{f^+ (\lambda)} & 0 \end{pmatrix} & \lambda \in (\lambda_{2n-2}, \lambda_{2n-1}) \\ (-1)^{n-1} e^{2i \sigma_3 \sqrt{\lambda} x + 8i \sigma_3 \sqrt{\lambda}^3 t} & \lambda \in (\lambda_{2n-1}, \lambda_{2n}) \end{cases} \]
Riemann–Hilbert Problem

For $x, t \in \mathbb{R}$ find a $2 \times 2$ matrix valued function $\Phi(x, t, \lambda)$ such that:

1. $\Phi(x, t, \lambda)$ is a holomorphic function of $\lambda$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}^+$.
2. $\Phi_\pm(x, t, \lambda)$ are continuous functions of $\lambda \in \mathbb{R}^+ \setminus \{\lambda_k\}_{k=0}^\infty$ that have at worst quartic root singularities on $\{\lambda_k\}_{k=0}^\infty$.
3. $\Phi_\pm(x, t, \lambda)$ satisfy the jump relation
   \[ \Phi_+(x, t, \lambda) = \Phi_-(x, t, \lambda) V(x, t, \lambda). \]
4. $\Phi(x, t, \lambda)$ has an asymptotic description of the form
   \[ \Phi(x, t, \lambda) = \begin{pmatrix} 1 & 1 \\ -i \sqrt{\lambda} & i \sqrt{\lambda} \end{pmatrix} \left( I + O\left(\sqrt[4]{\lambda}^{-1}\right) \right) B(\lambda) \] (7)
   as $\lambda \to \infty$ with $\lambda$ restricted to the complex plane with a sector of angle $0 < \theta < \frac{\pi}{4}$ around the positive real semiaxis removed.
5. There exist positive constants $c$, and $M$ such that
   \[ |\phi_{ij}(x, t, \lambda)| \leq Me^{c|\lambda|^2} \text{ for all } \lambda \in \mathcal{D}. \]

McLaughlin, Nabelek ’19
Theorem

Let \( x, t \in \mathbb{R} \) be fixed, then \( \Phi(x, t, \lambda) \) and \( \tilde{\Phi}(x, t, \lambda) \) solves our Riemann–Hilbert problem if and only if

\[
\tilde{\Phi}(x, t, \lambda) = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \Phi(x, t, \lambda)
\]

for some constant \( \alpha \). Moreover, the function \( u(x, t) \) determined from our Riemann–Hilbert problem by

\[
u(x, t) = 2i \frac{\partial}{\partial x} \lim_{\lambda \to \infty} \sqrt{\lambda} \left( 1 - \phi_{11}(x, t, \lambda) b_{11}(\lambda)^{-1} \right)
\]

solves the KdV equation.

The entries in the first row of \( \Phi(x, t, \lambda) e^{-i\sigma_3 \sqrt{\lambda} x} \) solve the Schrödinger equation

\[-\psi_{xx}(x, t, \lambda) + u(x, t)\psi(x, t, \lambda) = \lambda \psi(x, t, \lambda).\]
1. Consider spectral data is arbitrarily chosen to consist of, an increasing sequence $\{\lambda_k\}_{k=0}^{\infty}$, a sequence $\{\mu_n\}_{n=1}^{\infty}$ and a sequence $\{\sigma_n\}_{n=1}^{\infty}$ satisfying:
   ▶ $\lambda_0 < \lambda_1 < \cdots$, 
   ▶ $\mu_n \in (\lambda_{2n-1}, \lambda_{2n})$, 
   ▶ $R = \max_n |\lambda_{2n} - \lambda_{2n-1}| < \infty$, 
   ▶ for each $n$, $\sigma_n \in \{-1, 1\}$.
   ▶ There exists $N, C$ such that $E_n > Cn^2$ for all $n > N$.
   ▶ There exists $K$ such that the discs $D_n$ of radius $R$ centered at $(\lambda_{2n} + \lambda_{2n-1})/2$ are disjoint for $n \geq K$.

2. If the Riemann–Hilbert problem has a solution, then it is unique and produces a solution to the KdV equation as well as all solutions to the Schrödinger equation.

3. The existence theory will extend periodic KdV result to $L^\infty(\mathbb{R})$ periodic solution.
1. The solutions to the KdV equation produced in this manner have the trace form representation

\[ u(x, t) = \lambda_0 + \sum_{n=1}^{\infty} \lambda_{2n} + \lambda_{2n-1} - 2\mu_n(x, t) \]

where \( \mu_n(x, t) \) are the zeros of entries in \( \Phi(\lambda, x, t)B(\lambda)^{-1} \).

2. \( \mu_n(x, t) \) are the Dirichlet eigenvalues of \( L(t) = -\partial_x^2 + u(x, t) \) on \([x, x + T]\).

3. For an initial condition \( u(x) \in H_k([0, T]) \) periodically extended to the the whole line, \( (\lambda_{2n} - \lambda_{2n-1}) = O(n^{-k}) \).

4. For a square wave initial condition \( (\lambda_{2n} - \lambda_{2n-1}) = O(n^{-1}) \).
Theorem

If there exists a function $r_1(\lambda)$ such that

1. $r_1$ is holomorphic in $\mathbb{C} \setminus \sigma(L)$ with continuous boundary values $r_1 \pm$ on $\sigma(L)$ from above and below,
2. $r_1$ satisfies the jump relation $r_1^+(\lambda) = r_1^-(\lambda)^{-1}$ for $\lambda \in \sigma(L)$,
3. $r_1$ satisfies the asymptotic condition
   \[
   r_1(\lambda) = e^{i \sqrt{\lambda} T_1} (1 + O(\sqrt{\lambda})^{-1})
   \]  
   for $T_1 > 0$ and $\lambda$ restricted to the complex plane with a sector of angle $0 < \theta < \frac{\pi}{4}$ around the positive real semiaxis removed.
4. There exist positive constants $c$, and $M$ such that
   \[
   |r_1(\lambda)| \leq M e^{c|\lambda|^2}
   \]  
   for all $\lambda \in \mathcal{D}$.

then $u(x + T_1, t) = u(x, t)$. 


Theorem

If there exists a function $r_2(\lambda)$ such that

1. $r_2$ is holomorphic in $\mathbb{C} \setminus \sigma(L)$ with continuous boundary values $r_{2\pm}$ on $\sigma(L)$ from above and below,
2. $r_2$ satisfies the jump relation $r_{2+}(\lambda) = r_{2-}(\lambda)^{-1}$ for $\lambda \in \sigma(L)$,
3. $r_2$ satisfies the asymptotic condition
   \[ r_2(\lambda) = e^{4i\sqrt{\lambda}^3 T_2} (1 + O(\sqrt{\lambda}^{-1})) \quad (12) \]
   for $T_2 > 0$ and $\lambda$ restricted to the complex plane with a sector of angle $0 < \theta < \frac{\pi}{4}$ around the positive real semiaxis removed.
4. There exist positive constants $c$, and $M$ such that
   \[ |r_2(\lambda)| \leq Me^{c|\lambda|^2} \text{ for all } \lambda \in D. \]

then $u(x, t + T_2) = u(x, t)$. 

1. Consider a function $R_1(s)$ on $\Gamma = [-k_2, -k_1] \cup [k_1, k_2]$ such that $R_1(s) \geq 0$ for $s \in [k_1, k_2]$, $R_1(s) \leq 0$ for $s \in [-k_2, -k_1]$, and $R_1(s)$ is Hölder continuous on its support.

2. Suppose $f$ solves

$$f(s; x, t) - e^{-2sx+8s^3t} R_1(s) \mathcal{H}_\Gamma f(-s; x, t) = e^{-2sx+8s^3t} R_1(s),$$

$$\mathcal{H}_\Gamma f(s) = \int_\Gamma \frac{f(s')}{s - s'} ds'.$$

3. The function

$$u(x, t) = \frac{2}{\pi} \int_\Gamma \frac{\partial f}{\partial x}(s, x, t) ds$$

solves the KdV equation $u_t - 6uu_x + u_{xxx} = 0$.

4. For each fixed $t$, the Schrödinger operator $L = -\partial_x^2 + u(x, t)$ has spectrum

$$\sigma(L) = \{-s^2 : R_1(s) \neq 0\} \cup [0, \infty).$$
1. Any $g$ gap solution can be computed from choosing

$$R_1(s) = \exp \left( \sum_{j=1}^{g} s^{2j-1} \right) \sum_{\ell=1}^{g} \left( 1_{[\kappa_{2\ell-1}, \kappa_{2\ell}]}(s) - 1_{[-\kappa_{2\ell}, -\kappa_{2\ell-1}]}(s) \right)$$

and considering $\tilde{u}(x, t) = u(x - 6Ct, t) + C$.

2. The $g$ gap solutions are a dense subset of the periodic solutions (Marchenko, Ostrovskii ’75).

3. We can construct any $g$ gap solution (and thus any periodic solution) to the KdV equation as a limit of $N$ soliton solutions with respect to uniform convergence in compact sets.
Numerical N-Soliton Solutions
Primitive solution determined by:

\[ R_1(s) = e^{6s^5} \left[ \mathbb{1}_{\left[\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}}\right]}(s) + \mathbb{1}_{\left[-\frac{1}{\sqrt{6}}, \frac{\sqrt{2}}{\sqrt{3}}\right]}(s) + \mathbb{1}_{\left[\frac{\sqrt{5}}{\sqrt{6}}, 1\right]}(s) - \mathbb{1}_{\left[-1, -\frac{\sqrt{5}}{\sqrt{6}}\right]}(s) - \mathbb{1}_{\left[-\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{2}}\right]}(s) - \mathbb{1}_{\left[-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}}\right]}(s) \right] \]
Primitive Solutions with $R_1(s) = R(\mathbb{1}_{[\frac{1}{8},1]}(s) - \mathbb{1}_{[-1,-\frac{1}{8}]}(s))$

$R = 100$ (top) and $R = 0.01$ (bottom)
Three examples of 2-gap primitive potentials.

**Top:**

\[
R_1(s) = \mathbb{1}_{[\frac{1}{2}, \frac{1}{\sqrt{2}}]}(s) + \mathbb{1}_{[\frac{\sqrt{3}}{2}, 1]}(s) \\
- \mathbb{1}_{[-1, -\frac{\sqrt{3}}{2}]}(s) - \mathbb{1}_{[-\frac{1}{\sqrt{2}}, -\frac{1}{2}]}(s)
\]

**Middle:**

\[
R_1(s) = \mathbb{1}_{[\frac{1}{2}, \frac{1}{\sqrt{2}}]}(s) + \mathbb{1}_{[-\frac{\sqrt{3}}{2}, 1]}(s)
\]

**Bottom:**

\[
R_1(s) = 100\left[\mathbb{1}_{[\frac{1}{2}, \frac{1}{\sqrt{2}}]}(s) + \mathbb{1}_{[\frac{\sqrt{3}}{2}, 1]}(s) \\
- \mathbb{1}_{[-1, -\frac{\sqrt{3}}{2}]}(s) - \mathbb{1}_{[-\frac{1}{\sqrt{2}}, -\frac{1}{2}]}(s)\right]
\]
1. The following 2+1D system is completely integrable

\[ s_t + \frac{1}{4} s_u^2 + \frac{1}{4} s_v^2 + \Pi = 0 \]
\[ c_t + \frac{1}{2} (s_u c)_u + \frac{1}{2} (s_v c)_v - a \frac{1}{2} Ps + \frac{1}{8} Ps_{uv} = 0 \]

where \( P = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \), and \( \Pi \) is determined from \( c \) by \( \Pi_{uv} = \frac{1}{2} Pc \).
The Nonlocal $\bar{\partial}$ Problem

1. Consider the function $\chi$ solving the nonlocal $\bar{\partial}$ problem

$$\frac{\partial \chi}{\partial \lambda}(\lambda) = \int_{C} R_{0}(\lambda, \eta) e^{\phi(\eta)-\phi(\lambda)} \chi(\eta) dA(\eta)$$

with normalization $\chi(\lambda) \to 1$ as $\lambda \to \infty$, where $R_{0}$ is a generalized function with compact support on $\mathbb{C} \setminus 0$ and

$$\phi(\lambda, u, v, t) = \lambda u + a\lambda^{-1}v + \frac{1}{4}(\lambda^2 - \lambda^{-2})t.$$

2. The function $\chi(\lambda)$ has expansions

$$\chi(\lambda) = 1 + \chi_{1}^{\infty} \lambda^{-1} + O(\lambda^{-2}), \quad \chi(\lambda) = \chi_{0}^{0} + O(\lambda)$$

3. The functions $s$ and $c$ given by

$$s(u, v, t) = -\log(\chi_{0}^{0}(u, v, t))$$

$$c(u, v, t) = -(\chi_{1}^{\infty})_{v}(u, v, t) - \frac{1}{2} s_{uv}(u, v, t)$$

solve the 2+1D integrable generalization of the Kaup–Broer system.
If we look for a $u + v$ invariant solution, and define $\eta$ and $\varphi$ via

$$\sigma s(u, v, \sigma t) = \varphi(u + v, t), \quad -ac(u, v, \sigma t) = \eta(u + v, t),$$

then $\varphi$ and $\eta$ solve

$$\eta_t + \varphi_{xx} + (\eta\varphi_x)_x + \mu\varphi_{xxxx} = 0,$$
$$\varphi_t + \frac{1}{2}(\varphi_x)^2 + \varepsilon\eta = 0,$$

where $\mu = -a\sigma^2$ and $\varepsilon = -a/4$. Setting $a = \pm 1$ and $\sigma = 1$, i.e., exhausting all 4 canonical scalings of the 1+1D Kaup–Broer system.
1. Consider a function $R_1(s)$ on
\[ \Gamma = [\gamma_1, -\gamma_2] \cup [-\gamma_1, -\gamma_2] \cup [\gamma_1, \gamma_2] \] such that $R_1(s) \geq 0$ for $s \in [-\gamma_1, -\gamma_2] \cup [\gamma_1, \gamma_2]$, $R_1(s) \leq 0$ for $s \in [-\gamma_2, -\gamma_1] \cup [\gamma_2, \gamma_1]$, and $R_1(s)$ is Hölder continuous on its support.

2. Suppose $f$ solves
\[ f(s; x, t) - e^{\phi(s, x, t)} R_1(s) \mathcal{H}_\Gamma f(s^{-1}; x, t) = e^{\phi(s, x, t)} R_1(s), \]
\[ \phi(s, x, t) = -(s - s^{-1})x - \frac{1}{2}(s^2 - s^{-2})t. \]

3. Then the functions
\[ \begin{align*}
\varphi(x, t) &= -\log \left( 1 - \frac{1}{\pi} \int_{\Gamma} \frac{f(s; x, t)}{s} ds \right) \\
\eta(x, t) &= -\frac{1}{2} \varphi_{xx}(x, t) - \frac{1}{\pi} \int_{\Gamma} f_s(s; x, t) ds
\end{align*} \]
solve the 1+1D Kaup–Broer system ($a = -1$, $\sigma = 1$).
Numerical N-Soliton Solutions

Functions of the form $R_1(s) = e_1(s)f_1(s)$ where

$$e_1(s) = \exp\left(-2 \sum_{n=1}^{g} t_n(s^n - s^{-n})\right), \quad (14)$$

$$f_1(s) = \sum_{n=1}^{g_\ell} \left( \mathbb{1}_{[\eta_{2n-1},\eta_{2n}]}(s) - \mathbb{1}_{[\eta_{2n-1}^{-1},\eta_{2n-1}^{-1}]}(s) \right)$$

$$+ \sum_{m=1}^{g_r} \left( \mathbb{1}_{[-\xi_{2m-1}^{-1},-\xi_{2m}^{-1}]}(s) - \mathbb{1}_{[-\xi_{2m},-\xi_{2m-1}]}(s) \right)$$

give finite gap solutions.
Time reversal of primitive solution determined by

\[ R_1(s) = \pi e^{-10} \left( -\mathbb{I}_{[-2, -\frac{15}{8}]}(s) - \mathbb{I}_{[-\frac{7}{4}, -\frac{13}{8}]}(s) - \mathbb{I}_{[-\frac{11}{8}, -\frac{5}{4}]}(s) + \mathbb{I}_{[\frac{11}{8}, \frac{13}{8}]}(s) + \mathbb{I}_{[\frac{7}{4}, 2]}(s) \right). \]
Motivation: Soliton Gas

Rador et. al '19