Dispersive Shockwaves and Related Nonlinear Waves in Shallow Water

Patrik Vaclav Nabelek
In collaboration with Ken McLaughlin (CSU), Vladimir Zakharov (UA,LITP), Dmitry Zakharov (CMU), Sergey Dyachenko (UIUC)

November 16, 2018
The First Sighting of a Soliton

“I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on ..., preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it ...”

– John Scott Russel, 1844
1 Introduction and Background

2 The Cauchy Problem for the KdV Equation with a Periodic Initial Condition

3 Solution to Completely Integrable PDEs Via the Nonlocal $\bar{\partial}$ Problem
   - Solutions to the KP and KdV Equation
   - Solutions to the Kaup–Broer System
Completely integrable Hamiltonian systems

A Hamiltonian system with a $N$ dimensional configuration space is completely integrable if there exist $N$ Poisson commuting conservation laws in phase space.

Important finite dimensional integrable systems include

1. Motion in a central potential.
2. The spherical pendulum.
3. The Euler top equation.
4. The Kovalevskaya top equation

More recently, many infinite dimensional completely integrable systems have been discovered. Examples include

1. The Korteweg–de Vreis (KdV), Kadomtsev–Petviashvili (KP), and the Kaup–Broer (KB) system.
2. The cubic 1+1 dimensional nonlinear Schrödinger equations, the Davey–Stewardson equation, the Toda lattice.
The Kaup–Broer System

1. The non-dimensional KB system is

\[
\begin{align*}
\eta_t + \varphi_{xx} + (\eta \varphi_x)_x + \frac{1}{4} \varphi_{xxxx} &= 0 \\
\varphi_t + \frac{1}{2} (\varphi_x)^2 + \eta &= 0
\end{align*}
\]  \hspace{1cm} (1)

and is an approximation that holds when the ratios amplitude/depth and depth/wavelength are small, and bottom effects are considered thru a Boussinesq type approximation.

2. Considering the change of variables to \( r \) and \( s \) satisfying \( \eta = r + s \) and \( \varphi_x = r - s \) if the overlap between \( r \) and \( s \) is small they satisfy left and right moving KdV equations

\[
\begin{align*}
rt + rx + 2rr_x + \frac{1}{4} r_{xxx} &= 0, \\
st - sx - 2ss_x - \frac{1}{4} s_{xxx} &= 0.
\end{align*}
\]  \hspace{1cm} (2) (3)

3. The non-dimensional KdV equation we will use is

\[
u_t - 6uu_x + u_{xxx} = 0.
\]  \hspace{1cm} (4)
1. Many interesting solutions to the KdV equation can be interpreted as nonlinear superpositions of traveling wave solutions.

2. The traveling wave profile satisfy the equations of a degree 3 anharmonic oscillator.

3. The soliton solutions are given by

\[ u(x, t) = -\frac{c}{2} \text{sech}^2 \left( \sqrt{\frac{c}{4}} (x - ct - x_0) \right). \]  \hspace{1cm} (5)

4. The periodic cnoidal wave solutions are given by

\[ u(x, t) = 2 \wp(x - ct - x_0 + i\beta) - \frac{c}{6} \]  \hspace{1cm} (6)

where \( \wp \) is the Wierstrass elliptic function with half periods \( \alpha, i\beta \) and invariants \( g_2 = c^2/12 - A \) and \( g_3 = Ac/12 - c^3/216. \)
1 Introduction and Background

2 The Cauchy Problem for the KdV Equation with a Periodic Initial Condition

3 Solution to Completely Integrable PDEs Via the Nonlocal $\bar{\partial}$ Problem
   - Solutions to the KP and KdV Equation
   - Solutions to the Kaup–Broer System
1. Consider the system

\[ L\psi = E\psi, \quad \psi_t = A\psi \]  \hspace{1cm} (7)

where

\[ L = -\frac{\partial^2}{\partial x^2} + u(x,t), \]  \hspace{1cm} (8)

\[ A = -4\frac{\partial^3}{\partial x^3} + 6u(x,t)\frac{\partial}{\partial x} + 3u_x(x,t). \]  \hspace{1cm} (9)

2. The compatibility condition is the Lax equation

\[ L_t = [A, L]. \]  \hspace{1cm} (10)

3. This is isospectral and equivalent to

\[ u_t - 6uu_x + u_{xxx} = 0. \]  \hspace{1cm} (11)
1. The potential $u \in L^\infty(\mathbb{R})$ with $u(x + T) = u(x)$ is determined by:
   - The nondegenerate band ends $E_k$ such that the spectrum of $L$ breaks up as a disjoint union
     \[
     \sigma(L) = \bigcup_{j=0}^{G-1} [E_{2j}, E_{2j+1}] \cup [E_{2G}, \infty) \tag{12}
     \]
     where $G$ can be either finite or countably infinite. (WLOG $E_0 = 0$)
   - The Dirichlet eigenvalues $\mu_{n_k} \in [E_{2k-1}, E_{2k}]$.
   - The signature $\sigma_{n_k} = -\text{sgn}(\log(|y_2'(T, \mu_{n_k})|))$.

2. Because the KdV evolution is isospectral, it is equivalent to an evolution of $\mu_{n_k}, \sigma_{n_k}$. This evolution is described by the Dubrovin equation, and if $\mu_{n_k}$ and $E_j$ are known potential can be recovered from
   \[
   u(x, t) = E_0 + \sum_{k=1}^{G} E_{2k-1} + E_{2k} - 2\mu_{n_k}(x, t). \tag{13}
   \]
Riemann–Hilbert Problem

For $x, t \in \mathbb{R}$ find a $2 \times 2$ matrix valued function $\Phi(x, t, \lambda)$ such that:

1. $\Phi(x, t, \lambda)$ is a holomorphic function of $\lambda$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}^+$.

2. $\Phi_\pm(x, t, \lambda)$ are continuous functions of $\lambda \in \mathbb{R}^+ \setminus \{E_j\}_{j=0}^{2G}$ that have at worst quartic root singularities on $\{E_k\}_{k=0}^{2G} = 0$.

3. $\Phi_\pm(x, t, \lambda)$ satisfy the jump relation
   
   \[ \Phi_+(x, t, \lambda) = \Phi_-(x, t, \lambda)V(x, t, \lambda). \]

4. $\Phi(x, t, \lambda)$ has an asymptotic description of the form

   \[ \Phi(x, t, \lambda) = \begin{pmatrix} 1 & 1 \\ -i\sqrt{\lambda} & i\sqrt{\lambda} \end{pmatrix} \left( I + O\left(\sqrt{\lambda}^{-1}\right) \right) B(\lambda) \quad (14) \]

   as $\lambda \to \infty$ with $\lambda$ restricted to the complex plane with a sector of angle $0 < s < \frac{\pi}{8}$ around the real axis removed.

5. There exist positive constants $c$, and $M$ such that

   \[ |\phi_{ij}(x, t, \lambda)| \leq Me^{c|\lambda|^2} \text{ for all } \lambda \in \mathcal{D}. \]
Dependence on Auxiliary Data

Definition

\[
B(\lambda) := \frac{\sqrt{f^0(\lambda)}}{\sqrt[4]{\Delta(\lambda)^2 - 4}} \begin{pmatrix} f^-(\lambda) & 0 \\ 0 & f^+(\lambda) \end{pmatrix}, \tag{15}
\]

\[
f^\pm(\lambda) := \prod_{\sigma_n = \pm 1}^{\infty} \frac{T^2}{n^2\pi^2} (\mu_n - \lambda), \quad f^0(\lambda) := \prod_{\sigma_n = 0}^{\infty} \frac{T^2}{n^2\pi^2} (\mu_n - \lambda). \tag{16}
\]

\[
V(\lambda) := \begin{cases} 
(-1)^{k+m(\lambda)-1} \begin{pmatrix} 0 & i f^+(\lambda) \\ i f^-(\lambda) & 0 \end{pmatrix} & \lambda \in (E_{2k-2}, E_{2k-1}) \\
(-1)^{k+m(\lambda)-1} e^{2i\sigma_3\sqrt{\lambda}x} & \lambda \in (E_{2k-1}, E_{2k})
\end{cases} \tag{17}
\]

\[
m(\lambda) := |\{k \in \mathbb{N} : \mu_{n_k} \leq \lambda, \sigma_{n_k} = 0\}| \tag{18}
\]
Recovery of the KdV Solution

**Theorem**

Let \( x, t \in \mathbb{R} \) be fixed, then \( \Phi(x, t, \lambda) \) and \( \tilde{\Phi}(x, t, \lambda) \) solves our Riemann–Hilbert problem if and only if

\[
\tilde{\Phi}(x, t, \lambda) = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \Phi(x, t, \lambda)
\]

(19)

for some constant \( \alpha \). Moreover, the function \( u(x, t) \) determined from our Riemann–Hilbert problem by

\[
u(x, t) = 2i \frac{\partial}{\partial x} \lim_{\lambda \to \infty} \sqrt{\lambda} \left(1 - e^{i \sqrt{\lambda} x + 4i \sqrt{\lambda}^3 t} [\Phi(x, t, \lambda) B(\lambda)^{-1}]_{11}\right)
\]

(20)

solves the KdV equation.
Theorem

If there exists a function \( r_1(\lambda) \) such that

1. \( r_1 \) is holomorphic in \( \mathbb{C} \setminus \sigma(L) \) with continuous boundary values \( r_{1\pm} \) on \( \sigma(L) \) from above and below,
2. \( r_1 \) satisfies the jump relation \( r_{1+}(\lambda) = r_{1-}(\lambda)^{-1} \) for \( \lambda \in \sigma(L) \),
3. \( r_1 \) satisfies the asymptotic condition

\[
    r_1(\lambda) = e^{i\sqrt{\lambda}T_1} (1 + O(\sqrt{\lambda}^{-1}))
\]

for \( T_1 > 0 \) and \( \lambda \) restricted to the complex plane with a sector of angle \( 0 < s < \frac{\pi}{8} \) around the real axis removed.

4. There exist positive constants \( c, \) and \( M \) such that \( |r_1(\lambda)| \leq Me^{c|\lambda|^2} \) for all \( \lambda \in \mathcal{D} \).

then \( u(x + T_1, t) = u(x, t) \).
Spectral Periodicity Conditions in Time for KdV

**Theorem**

If there exists a function $r_2(\lambda)$ such that

1. $r_2$ is holomorphic in $\mathbb{C} \setminus \sigma(L)$ with continuous boundary values $r_{2\pm}$ on $\sigma(L)$ from above and below,
2. $r_2$ satisfies the jump relation $r_{2+}(\lambda) = r_{2-}(\lambda)^{-1}$ for $\lambda \in \sigma(L)$,
3. $r_2$ satisfies the asymptotic condition

\[
r_2(\lambda) = e^{4i\sqrt{\lambda}^3T_2}(1 + O(\sqrt{\lambda})^{-1})
\]  

(22)

for $T_2 > 0$ and $\lambda$ restricted to the complex plane with a sector of angle $0 < s < \frac{\pi}{8}$ around the real axis removed.

4. There exist positive constants $c$, and $M$ such that $|r_2(\lambda)| \leq Me^{c|\lambda|^2}$ for all $\lambda \in \mathcal{D}$.

then $u(x, t + T_2) = u(x, t)$. 

Patrik Vaclav Nabelek In collaboration with Ken McLaughlin (CSU), Vladimir Zakharov (UA,LITP), Dmitry Zakharov (CMU), Sergey Dyachenko (UIUC)

Dispersive Shockwaves

November 16, 2018 16 / 38
1 Introduction and Background

2 The Cauchy Problem for the KdV Equation with a Periodic Initial Condition

3 Solution to Completely Integrable PDEs Via the Nonlocal $\bar{\partial}$ Problem
   - Solutions to the KP and KdV Equation
   - Solutions to the Kaup–Broer System
The Lax Pair for the KP Equation

1. We now consider the more general linear system

\[ L\psi = 0, \quad \psi_t = A\psi, \quad (23) \]

\[ L = -\frac{\partial^2}{\partial x^2} - \alpha \frac{\partial}{\partial y} + u(x, y, t), \quad (24) \]

\[ A = -4 \frac{\partial^3}{\partial x^3} + 6u(x, y, t) \frac{\partial}{\partial x} + 3w(x, y, t). \quad (25) \]

2. The compatibility condition is the Lax equation

\[ L_t = [A, L]. \quad (26) \]

3. This is equivalent to a system involving \( u \) and \( w \) that reduces to

\[ (u_t - 6uu_x + u_{xxx})_x + 3\alpha^2 u_{yy} = 0. \quad (27) \]
Deformed Lax Operators

1. Introduce the differential operators

\[ D_x = \frac{\partial}{\partial x} + i\lambda, \quad D_y = \frac{\partial}{\partial y} + \alpha\lambda^2, \quad D_t = \frac{\partial}{\partial t} + 4i\lambda^3, \quad (28) \]

\[ L_1 = -D_x^2 - \alpha D_y + u(x, y, t), \quad (29) \]

\[ L_2 = -4D_x^3 + 6u(x, y, t)D_x + 3w - D_t. \quad (30) \]

2. If \( \chi \) satisfies \( L_j \chi = 0 \) and \( \psi = e^{\phi(\lambda)}\chi \) in terms of

\[ \phi(\lambda) = i\lambda x + \alpha\lambda^2 y + 4i\lambda^3 t \quad (31) \]

then \( L\psi = 0, \psi_t = A\psi. \)

3. The compatibility condition for the Lax equation is the KP equation, so if \( L_j \chi = 0 \) then \( u \) solves the KP equation.
1. If $\lambda = x + iy$, then

$$\frac{\partial}{\partial \lambda} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{\lambda}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

(32)

2. The operator $\frac{\partial}{\partial \lambda}$ is the usual complex differential.

3. The equation $\frac{\partial f}{\partial \lambda} = 0$ is the Cauchy–Riemann equations.
The Nonlocal $\bar{\partial}$ Problem

1. Determine $\chi$ as the solution to the nonlocal $\bar{\partial}$ problem

$$\frac{\partial \chi}{\partial \lambda}(\lambda) = \int_{\mathbb{C}} R_0(\lambda, \eta)e^{\phi(\eta) - \phi(\lambda)} \chi(\eta)dA(\eta)$$

(33)

normalized so that $\chi \to 1$ as $\lambda \to \infty$, or the equivalent integral equation

$$\chi(\lambda) = 1 + \frac{1}{\pi} \int_{\mathbb{C}} \int_{\mathbb{C}^2} \frac{R_0(\xi, \eta)e^{\phi(\eta) - \phi(\xi)} \chi(\eta)}{\lambda - \xi} dA(\eta)dA(\xi).$$

(34)

2. Equation (33) involves the spectral parameter $\lambda$, it is a generalization of the analytical characterization of the Jost solutions appearing in general spectral problems.

3. Equation (33) and its normalization produces a function $\chi$, but $u$ and $w$ in $L_j$ have yet to be defined. The goal is to define $u$ and $w$ in terms of $\chi$ so that the compatibility condition $L_j \chi(\lambda) = 0$ is satisfied for all $\lambda$. 

Patrik Vaclav Nabelek In collaboration with Ken McLaughlin (CSU), Vladimir Zakharov (UA,LITP), Dmitry Zakharov (CMU), Sergey Dyachenko (UIUC)

Dispersive Shockwaves

November 16, 2018 21 / 38
The Nonlocal $\bar{\partial}$ Problem for $L_j \chi$

1. The key-point is that the choice of $\phi$ and $D_{x,y,t}$ used to define $L_j$ have been defined in such a way that $L_j \chi(\lambda)$ also solve the nonlocal $\bar{\partial}$ problem

$$\frac{\partial (L_j \chi)}{\partial \lambda}(\lambda) = \int_\mathbb{C} R_0(\lambda, \eta)e^{\phi(\eta)-\phi(\lambda)}(L_j \chi)(\eta)dA(\eta).$$ (35)

2. With the choices

$$u = 2i\chi_{0x}, \quad w = 4i\chi_{0xx} - 4\chi_{1x} - 2iu\chi_0$$ (36)

we have $L_j \chi \to 0$ as $\lambda \to \infty$.

3. Then $L_j \chi$ solves the homogenous integral equation

$$L_j \chi(\lambda) = \frac{1}{\pi} \int_\mathbb{C} \int_\mathbb{C} \frac{R_0(\xi, \eta)e^{\phi(\eta)-\phi(\xi)}\chi(\eta)}{\lambda - \xi}dA(\eta)dA(\xi)$$ (37)

and therefore $L_j \chi \equiv 0$. 
1. The specific choice of

\[ R_0(\lambda, \eta) = \pi \sum_{n=1}^{N} c_n \delta(\lambda - k_n) \delta(\eta - w_n). \]  

allows us to produce N–soliton solutions.

2. The \( \bar{\partial} \) problem is then solved by

\[ \chi(\lambda) = 1 + \sum_{n=1}^{N} \frac{\chi_n(x, y, t)}{\lambda - k_n}, \]  

\[ \sum_{n=1}^{N} \left( \delta_{mn} + \frac{c_m e^{\phi(w_m) - \phi(k_m)}}{k_n - w_m} \right) \chi_n = c_m e^{\phi(w_m) - \phi(k_m)}. \]  

3. The N–soliton solution is given by

\[ u(x, y, t) = 2i \frac{\partial}{\partial x} \sum_{n=1}^{N} \chi_n. \]
1. The choice \( R_0(\lambda, \eta) = \tilde{R}_0(\lambda)\delta(\lambda + \eta) \) produces \( \chi \) that is independent of \( y \). In terms of \( N \)-solitons this amounts to taking \( w_m = -k_m \) so that \( e^{\phi(w_m) - \phi(k_m)} = e^{-2i k_m x - 8i k_m^3 t} \).

2. At each time \( t \), the potential \( u(x, t) \) leads to \( L = -\frac{\partial^2}{\partial x^2} + u(x, t) \) with a reflectionless continuous spectrum \( [0, \infty) \), and discrete spectrum \( \{k_1^2, k_2^2, \ldots, k_N^2\} \).
Video
1. Consider spectral parameters filling $-i\Gamma$, $\Gamma = \left[ \frac{3}{8}, \frac{5}{8} \right] \cup \left[ \frac{3}{4}, 1 \right]$, and scale the connection coefficients so as $N \to \infty$ the linear system for the N-solitons limits to

$$f(s; x, t) - \pi e^{-2sx+8s^3t} \mathcal{H}_\Gamma f(-s; x, t) = e^{-2sx+8s^3t},$$

and the function $\chi$ is of the form

$$\chi(\lambda; x, t) = 1 + \int_\Gamma \frac{f(s; x, t)}{\lambda - s} ds.$$  

(42)

(43)

2. The spectrum of $L = -\partial_x^2 + u(x, t)$ at each time is a simple continuous spectrum on $[-1, -\frac{9}{8}] \cup [-\frac{25}{64}, -\frac{3}{8}] \cup [0, \infty)$.

3. The function $\chi = [\chi(\lambda), \chi(-\lambda)]$ solves a Riemann–Hilbert problem with jump condition

$$\chi_+(s) = \chi^+(s) = \begin{cases} 
\chi^-(s) \begin{pmatrix} 1 & 0 \\ 2ie^{-2sx+8s^3t} & 1 \end{pmatrix} & s \in \Gamma \\
\chi^-(s) \begin{pmatrix} 1 & 2ie^{2sx-8s^3t} \\ 0 & 1 \end{pmatrix} & s \in \Gamma^{-1}
\end{cases}.$$  

(44)
1 Introduction and Background

2 The Cauchy Problem for the KdV Equation with a Periodic Initial Condition

3 Solution to Completely Integrable PDEs Via the Nonlocal \( \bar{\partial} \) Problem
   - Solutions to the KP and KdV Equation
   - Solutions to the Kaup–Broer System
1. We now consider the system

\[\begin{align*}
    s_t + \frac{1}{4} \left( s_u^2 - s_{uu} \right) + \frac{1}{4} \left( s_v^2 - s_{vv} \right) + \rho &= 0 \\
    B_t + \frac{1}{4} (B_{uu} + 2(s_u B)_u + 2s_{uu}) &+ \frac{1}{4} (B_{vv} + 2(s_v B)_v - 2as_{vv}) = 0, \\
    \rho_{uv} &= \frac{1}{2} B_{uu} + \frac{1}{2} B_{vv}.
\end{align*}\] (45, 46, 47, 48)

2. This system is equivalent to

\[\begin{align*}
    L\psi &= 0, \quad \psi_t = M\psi \implies L_t &= [M, L] + QL, \\
    L &= \frac{\partial^2}{\partial u \partial v} + A \frac{\partial}{\partial v} + B + 1, \\
    M &= \frac{1}{4} \frac{\partial^2}{\partial u^2} + \frac{1}{4} \frac{\partial^2}{\partial v^2} + F \frac{\partial}{\partial v} + G, \\
    A_v &= -2F_u = s_{uv}, \quad G_v = \frac{1}{2} B_u, \quad Q = F_v - \frac{1}{2} A_u.
\end{align*}\] (49, 50, 51, 52)
1. We again consider

\[ \frac{\partial \chi}{\partial \bar{\lambda}}(\lambda) = \int_{\mathcal{C}} R_0(\lambda, \eta) e^{\phi(\eta) - \phi(\lambda)} \chi(\eta) dA(\eta). \] (53)

2. This time we consider

\[ \phi = \lambda u + \frac{a}{\lambda} v + \frac{1}{4} \left( \lambda^2 - \frac{1}{\lambda^2} \right) t. \] (54)

3. Choose the operators

\[ D_u = \frac{\partial}{\partial u} + \lambda, \quad D_v = \frac{\partial}{\partial v} + \frac{a}{\lambda}, \quad D_t = \frac{\partial}{\partial t} + \frac{1}{4} \left( \lambda^2 - \frac{1}{\lambda^2} \right). \] (55)
1. We choose the operators

\[
L_1 \chi = D_u D_v \chi + A D_v \chi + (B - a) \chi
\]

\[
= \chi_{uv} + \lambda \chi_v + \frac{a}{\lambda} \chi_u + A \chi_v + A \frac{a}{\lambda} \chi + B,
\]

(56)

\[
L_2 \chi = D_u D_v \chi + A D_v \chi + (B - a) \chi
\]

\[
= \chi_{uv} + \lambda \chi_v + \frac{a}{\lambda} \chi_u + A \chi_v + A \frac{a}{\lambda} \chi + B.
\]

(57)

\[
L_2 \chi = D_u D_v \chi + A D_v \chi + (B - a) \chi
\]

(58)

\[
\frac{\partial L_j \chi}{\partial \lambda} (\lambda) = \gamma_j \delta(\lambda) + \int_\mathbb{C} R_0(\lambda, \eta) e^{\phi(\eta) - \phi(\lambda)} L_j \chi(\eta) dA(\eta),
\]

(60)

\[
\gamma_1 = \pi a (\chi_u(0) + A \chi(0)), \quad \gamma_2 = -\pi a (2 \beta \chi_v(0) + F \chi(0))
\]

(61)
Showing $L_j \chi \equiv 0$

1. This time, to guarantee $L_j \chi \equiv 0$ we need to check both $\gamma_j = 0$ in addition to $L_j \chi \to 0$ as $\lambda \to \infty$.

2. From the $\bar{\partial}$ inversion formula

$$L_j \chi(\lambda) = \frac{\gamma_j}{\lambda} + \frac{1}{\pi} \iint_{\mathbb{C}^2} \frac{R_0(\xi, \eta)e^{\phi(\eta)-\phi(\xi)} \chi(\eta)}{\lambda - \xi} dA(\eta)dA(\xi) \quad (62)$$

we see that $\gamma_j = 0$ is equivalent to the condition that $L_j \chi$ is regular at 0.

3. Explicitly, these conditions are satisfied if

$$A = s_u, \quad F = 2\beta s_v, \quad B = -\chi_0 v, \quad G = -2\alpha \chi_0 u \quad (63)$$

where $s = -\log(\chi(0))$ and $\chi(\lambda) = 1 + \frac{\chi_0}{\lambda} + O(\lambda^{-2})$. 
1. To produce $N$–solitons choose

$$R_0(\lambda, \eta) = \pi \sum_{n=1}^{N} c_n \delta(\lambda - z_n) \delta(\eta - w_n). \quad (64)$$

2. The $\bar{\partial}$ problem is then solved by

$$\chi(\lambda) = 1 + \sum_{n=1}^{N} \frac{\chi_n(u, v, t)}{\lambda - z_n}, \quad (65)$$

$$\sum_{n=1}^{N} \left( \delta_{mn} + \frac{c_m e^{\phi(w_m)} - \phi(z_m)}{z_n - w_m} \right) \chi_n = c_m e^{\phi(w_m)} - \phi(z_m). \quad (66)$$

3. The $N$–soliton solution is given by

$$B(u, v, t) = -\frac{\partial}{\partial v} \sum_{n=1}^{N} \chi_n, \quad s(u, v, t) = -\log \left( 1 - \sum_{n=1}^{N} \frac{\chi_n}{z_n} \right). \quad (67)$$
Reduction to 1+1 Dimensions

1. If $R_0(\lambda, \eta) = \delta(\lambda - \eta^{-1}) \tilde{R}_0(\lambda)$ then $\chi$ is $y$ independent.
2. In terms of the 2+1 dimensional N–solitons, this reduction amounts to taking $w_n = z_n^{-1}$.
3. Then $\varphi = s$ and $\eta = B - \frac{1}{2} s_{xx}$ satisfy the non dimensional Kaup–Broer system

\begin{align*}
\varphi_t + \frac{1}{2} \varphi^2_x + \eta &= 0 \quad (68) \\
\eta_t + \varphi_{xx} + (\eta \varphi_x)_x + \frac{1}{4} \varphi_{xxxx} &= 0. \quad (69)
\end{align*}
1. Consider spectral parameters filling $\Gamma_- = [-\frac{4}{5}, -\frac{4}{7}]$ and $\Gamma_+ = [\frac{1}{2}, \frac{2}{3}]$ in such a way so that as $N \to \infty$ the linear equation for the N-soliton solution becomes

$$\begin{align*}
\begin{cases}
f(s; x, t) + \pi e^{\phi(s^{-1})-\phi(s)} \mathcal{H}_\Gamma f(s^{-1}; x, t) = -e^{\phi(s^{-1})-\phi(s)} & s \in \Gamma_+ \\
f(s; x, t) - \pi e^{\phi(s^{-1})-\phi(s)} \mathcal{H}_\Gamma f(s^{-1}; x, t) = e^{\phi(s^{-1})-\phi(s)} & s \in \Gamma_-.
\end{cases}
\end{align*}$$

(70)

2. The function $\chi(\lambda) = [\chi(\lambda), \chi(\lambda^{-1})]$ solves a Riemann–Hilbert problem with jump

$$\chi^+(s) = \begin{cases}
\chi^-(s) \begin{pmatrix} 1 & 0 \\ \pm 2ie^{\phi(s^{-1})-\phi(s)} & 1 \end{pmatrix} & s \in \Gamma \\
\chi^-(s) \begin{pmatrix} 1 & \mp 2ie^{\phi(s^{-1})-\phi(s)} \\ 0 & 1 \end{pmatrix} & s \in \Gamma^{-1}
\end{cases}.$$ 

(71)
Future Work

1. Regularize the integral equations!
2. Use the Riemann–Hilbert problem to analyze specific periodic initial condition to the KdV equation.
3. Prove spatial and longtime asymptotic results for the limiting solution of the KdV and KB system by adapting the method of [GGM18].
4. Consider infinite soliton limits where the spectral parameters are drawn from a probability distribution.
5. Produce 2+1 dimensional dispersive shockwave type solutions to the KP equation using the nonlocal \( \bar{\partial} \) problem.

Figure: A numerical 2+1D dispersive shockwave solution at a fixed time to the full shallow water potential Euler system with constant bathymetry computed by Jeffrey Knowels (PhD candidate in civil and construction engineering).